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down with determinants (?)

Posted By [lieven](#) On May 18, 2007 @ 12:15 In [math](#) | [8 Comments](#)



The [1] [Down with determinants!](#). Here is 'his' proof of the fact that any n by n matrix A has at least one eigenvector. Take a vector $v \in \mathbb{C}^n$, then as the collection of vectors $v, A.v, A^2.v, \dots, A^n.v$ must be linearly dependent, there are complex numbers $a_i \in \mathbb{C}$ such that

$$(a_0 + a_1 A + a_2 A^2 + \dots + a_n A^n).v = \vec{0} \in \mathbb{C}^n$$

But then as \mathbb{C} is algebraically closed the polynomial on the left factors into linear factors

$$a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = c(x - r_1)(x - r_2) \dots (x - r_n)$$

and therefore as $c(A - r_1 I_n)(A - r_2 I_n) \dots (A - r_n I_n).v = \vec{0}$ from which it follows that at least one of the linear transformations

$$A - r_j I_n$$

has a non-trivial kernel, whence A has an eigenvector with eigenvalue r_j .

Okay, fine, nice even, but does this simple minded observation warrant the extreme conclusion of his paper (on page 18) ?

As mathematicians, we often read a nice new proof of a known theorem, enjoy the different approach, but continue to derive our internal understanding from the method we originally learned. This paper aims to change drastically the way mathematicians think about and teach crucial aspects of linear algebra.

The simple proof of the existence of eigenvalues given in Theorem 2.1 should be the one imprinted in our minds, written on our blackboards, and published in our textbooks. Generalized eigenvectors should become a central tool for the understanding of linear operators. As we have seen, their use leads to natural definitions of multiplicity and the characteristic polynomial. Every mathematician and every linear algebra student should at least remember that the generalized eigenvectors of an operator always span the domain (Proposition 3.4)—this crucial result leads to easy proofs of upper-triangular form (Theorem 6.2) and the Spectral Theorem (Theorems 7.5 and 8.3).

Determinants appear in many proofs not discussed here. If you scrutinize such proofs, you'll often discover better alternatives without determinants. Down with Determinants!

I welcome all new proofs of known results as they allow instructors to choose the one best suited to their students (and preferable giving more than one proof showing that there is no such thing as 'the best way' to prove a mathematical result). What worries me is Axler's attitude shared by extremists and dogmatics world-wide : they are so blinded by their own right that they impoverish their own lives (and if they had their way, also that of others) by not willing to consider other alternatives.

A few other comments :

1. I would be far more impressed if he had given a short argument for the one line he skates over in his proof, that of \mathbb{C} being algebraically closed. Does anyone give a proof of this fact anymore or is this one of the few facts we expect first year students to accept on faith?
2. I don't understand this aversity to the determinant (probably because of its nonlinear character) but at the same time not having any problems with successive powers of matrices. Surely he knows that the determinant is a fixed \mathbb{Q} -polynomial in the traces (which are linear!) of powers of the matrix.
3. The essence of linear algebra is that by choosing a basis cleverly one can express a linear operator in an extremely nice matrix form (a canonical form) so that all computations become much more easy. This crucial idea of considering different bases and their basechange seems to be missing from Axler's approach. Moreover, I would have thought that everyone would know these days that 'linear algebra done right' is a well developed topic called 'representation theory of quivers' but I realize this might be viewed as a dogmatic statement. Fortunately someone else is giving the basic linear algebra courses here in Antwerp so students are spared my private obsessions (at least the first few years...).

In [2] [THE rationality problem](#) and which may be well known to others.

Recall from the previous post that rationality of the quotient variety of matrix-couples

$$(A, B) \in M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) / GL_n$$

under *simultaneous conjugation* is a very hard problem. On the other hand, the 'near miss' problem of the quotient variety of matrix-couples

$$(A, B) \quad \det(A) = 0 / GL_n$$

is completely trivial. It is rational for all n . Here is a one-line proof. Consider the quiver



then the dimension vector $(n-1, n)$ is a Schur root and the first fundamental theorem of GL_n (see for example Hanspeter Kraft's excellent book on invariant theory) asserts that the corresponding quotient variety is the one above. The result then follows from Aidan Schofield's paper [3] [Birational classification of moduli spaces of representations of quivers](#).

Btw. in this special case one does not have to use the full force of Aidan's result. [4] [Atlanta](#), emailed the following elegant short proof

Here is an outline of a geometric proof. Let $X = (A, B) : \det(A) = 0 \subset M_n^2$ and $Y = \mathbb{P}^{n-1} \times M_n$. Applying the no-name lemma to the PGL_n -equivariant dominant rational map

$$X \rightarrow Y \text{ given by } (A, B) \rightarrow (\text{Ker}(A), B)$$

(which makes X into a vector bundle over a dense open PGL_n -invariant subset of Y), we see that

$$X // PGL_n \text{ is rational over } Y // PGL_n$$

On the other hand, $Y // PGL_n = M_n // PGL_n$ is an affine space. Thus $X // PGL_n$ is rational.

The moment I read this I knew how to do this quiver-wise and that it is just another Brauer-Severi type argument so completely inadequate to help settling the genuine matrix-problem.

Update on the [5] [paper by Esther Beneish](#) : Esther did submit the paper in february.

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[1] Down with determinants!: <http://www.axler.net/DwD.html>

[2] THE rationality problem: <http://www.neverendingbooks.org/?p=318>

[3] Birational classification of moduli spaces of representations of quivers:

<http://www.arxiv.org/abs/math/9911014>

[4] Atlanta: <http://www.mathcs.emory.edu/~skip/conf/Home.html>

[5] paper by Esther Beneish: <http://www.arxiv.org/abs/0704.3450>

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