Socrates Intensive Program

Geometric and Algebraic Methods with Applications in Physics

University of Antwerp July 13th -July 27th 2008

Abelian ideals of Borel subalgebras and multiplets of representations

Paolo Papi

References

[1] R. Carter, *Lie algebras of finite and affine type*, Cambridge studies in advanced mathematics 96, Cambridge University Press.

[2] P. Cellini, P. Möseneder Frajria and P. Papi, *Abelian subalgebras in $\mathbb{Z}_2$-graded Lie algebras and affine Weyl groups*, Internat. Math. Res. Notices **43** (2004), 2281–2304.

[3] P. Cellini and P. Papi, *ad-nilpotent ideals of a Borel subalgebra*, J. Algebra **225** (2000), 130–141.

[4] R. Goodman, N. Wallach, *Representations and invariants of the classical groups.* Encyclopedia of Mathematics and its Applications, 68. Cambridge University Press

[5] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, 1978.

[6] J.-S. Huang and P. Pandžić, *Dirac cohomology, unitary representations and a proof of a conjecture of Vogan*, J. Amer. Math. Soc. **15** (2002), no. 1, 185–202 (electronic).

[7] J.-S. Huang and P. Pandžić, *Dirac operators in representation theory. Mathematics: Theory & Applications.* Birkhuser Boston, Inc., Boston, MA, 2006.

[8] J. E. Humphreys *Introduction to Lie algebras and representation theory.* Graduate Texts in Mathematics, Vol. 9.

[9] V. G. Kac, I. Todorov *Superconformal current algebras and their unitary representations.* Comm. Math. Phys. **102** (1985), no. 2, 337–347

[10] V. G. Kac, *Infinite dimensional Lie algebras*, third ed., Cambridge University Press, Cambridge, 1990.

[11] V. G. Kac, P. Möseneder Frajria and P. Papi, *Multiplets of representations, twisted Dirac operators and Vogan's conjecture in affine setting*, Adv. Math. (2008), **217** n.6, 2485–2562.

[12] B. Kostant, *Eigenvalues of the Laplacian and commutative Lie subalgebras.* Topology **3** (1965) suppl. 2, 147–159.

[13] B. Kostant, *Lie algebra cohomology and the generalized Borel-Weil theorem.* Ann. of Math. (2) **74** (1961), 329–387

[14] B. Kostant, *The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations.* Internat. Math. Res. Notices (1998), no. 5, 225–252.

[15] B. Kostant, *A cubic Dirac operator and the emergence of Euler number multiplets of representations for equal rank subgroups*, Duke Math. J. **100** (1999), no. 3, 447–501.

[16] B. Kostant, *A generalization of the Bott-Borel-Weil theorem and Euler number multiplets of representations.* Conference Mosh Flato 1999 (Dijon). Lett. Math. Phys. **52** (2000), no. 1, 61–78.

[17] B. Kostant, *Dirac cohomology for the cubic Dirac operator*, Studies in memory of Issai Schur (Chevaleret/Rehovot, 2000), Progr. Math., vol. 210, Birkhäuser Boston, 2003, pp. 69–93.

[18] B. Kostant *Powers of the Euler product and commutative subalgebras of a complex simple Lie algebra.* Invent. Math. **158** (2004), no. 1, 181–226.

[19] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[20] G. D. Landweber, *Multiplets of representations and Kostant's Dirac operator for equal rank loop groups*, Duke Math. J. **110** (2001), no. 1, 121–160.

[21] P. Möseneder Frajria, P. Papi *Casimir operators, abelian subspaces and $\mathfrak{u}$-cohomology*. Rend. Mat. Appl. (7) **27** (2007), no. 3-4, 265–276.

[22] D. I. Panyushev, *The exterior algebra and "spin" of an orthogonal $\mathfrak{g}$-module*. Transform. Groups **6** (2001), no. 4, 371–396.

[23] R Parthasarathy, *Dirac operators and the discrete series*. Ann. of Math. **96** (1972), no. 1, 1–30.

[24] N. R. Wallach, *Real reductive groups I*, Academic Press, 1988.

1 Background

Among the many textbooks, [8], [10] are standard references for finite dimensional and Kac-Moody Lie algebras, respectively. The monograph [1] is an introduction to both the finite and infinite dimensional theory, with emphasis on the affine case.

A complete and elementary treatment of Lie algebra cohomology, Clifford algebras and spin representations can be found in [4]; for the latter topics see also [7].

A standard reference for real reductive groups is Wallach's book [24]. Helgason [5] provides a comprehensive treatment of symmetric spaces.

Another useful (though less elementary) reference in Kumar's book [19]: the "Laplacian calculation" mentioned in the final lecture is taken from there.

2 Dirac operators and Vogan conjecture

The algebraic version of the Dirac operator has been introduced in [23]. My treatment is based on the first chapters of the monograph [7], which expand on the paper [6] to provide a treatment accessible to non-experts. I refer to chapters 1-4 of that book for the missing details in proofs and for the exercises. The subsequent chapters deal with applications of the Vogan conjecture to the theory of reductive groups.

3 References for Kostant's work

- $\mathfrak{u}$-cohomology: [13];
- $\mathfrak{g}$-module structure of $\wedge\mathfrak{g}$ and commutative subalgebras: [12];
- abelian ideals and Peterson's theorem:[14];

- cubic Dirac operator: [15];
- comparison between the module structure of $Ker\, D$ and Lie algebra cohomology: [16];
- proofs of old results on $\wedge\mathfrak{g}$ using affine $\mathfrak{u}$-cohomology: [18];
- Vogan conjecture for the cubic Dirac operator: [17].

4 Comments

My treatment of the proof of the old results by Kostants using $\mathfrak{u}$-cohomology follows [21], which for the results covered in the last two lectures is inspired by [18]. The proof of Peterson's theorem is borrowed from [3].

There is a kind of graded theory for abelian ideals of Borel subalgebras: Kostant's results from [12] were generalized to this setting by Panuyshev [22], whereas Peterson's encoding was extended in [2]. The treatment in [21] covers Panyushev's (and other people's) results using $\mathfrak{u}$-cohomology in affine setting.

An affine analog of the cubic Dirac operator was introduced in [9]. It has been used in [20] to obtain a multiplet decomposition in certain affine cases and thoroughly investigated in [11] (in the framework of vertex algebras), in order to establish an analog of the Vogan conjecture in affine setting.

Paolo Papi: Dipartimento di Matematica, Sapienza Università di Roma, P.le A. Moro 2, 00185, Roma , ITALY;
papi@mat.uniroma1.it