From Biplanes to the Klein quartic and the Buckyball

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Abstract

The Fano plane embedding in Klein's Riemann surface is constructed by an embedding of the 2-biplane in that surface. We show how an an embedding of the 3-biplane in a particular Riemann surface of genus 70, related to the Hecke group H^5, corresponds to embedding Buckyballs in that surface.

We also show that there is a 2-biplane structure on the cusps of the principal congruence subgroup $\Gamma(7)$, and a 3-biplane structure on the cusps of the principal congruence subgroup $H^5(4-\sqrt{5})$ of the Hecke group H^5.

1 Introduction

In 1995, Bertram Kostant [K] wrote a beautiful account that relates the truncated icosahedron to some ideas in the last letter of Galois that was written on the night before his fatal duel. Galois' result is that the group PSL(2, p) which acts transitively on the $p+1$ points of GF(p) $\cup \{\infty\}$ only acts transitively on p points if $p = 2, 3, 5, 7, 11$. Geometrically the most interesting case are $p = 7$ and $p = 11$. In this paper we discuss the many similarities between these two cases, giving links with the biplanes of orders 2 and 3 and embeddings of these planes into Riemann surfaces. We now briefly describe the connections with Riemann surfaces. The case $p = 7$ follows work in Klein's classic paper reprinted in [L1]. In Klein's quartic surface there are two classes of 7 embedded truncated cubes. We call the truncated cubes in one class *vertices* and the other class *lines*. If we consider their intersection pattern we get an embedding of the Fano plane in Klein's surface. This is a different method of obtaining the emdedding than found in [S2]. For $p = 11$, we find a Riemann surface of genus 70 that contains two classes of embedded truncated icosahedra (buckyballs.) In a similar way, we find an intersection pattern that defines a 3-biplane. We also discuss how the 2-biplane and 3-biplane can be constructed from the cusps of Fuchsian groups that define these surfaces. The following table presents some of the similarities that we have found for the two cases $p = 7$ and $p = 11$.

Property	$p = 7$	$p = 11$
Riemann surface	Klein's quartic of genus 3	A Riemann surface of genus 70
Automorphism group	PSL(2,7)	PSL(2,11)
Combinatorial structure	Fano plane and order 2- biplane	order 3-biplane
Platonic solids	Contains 7 imbedded cubes	contains 11 imbedded icosahedra
Truncations	truncated cubes	Buckyballs
Fuchsian group	Modular group Γ	Hecke group H^5
Triangle group	(2,3,7)	(2,5,11)
Modular interpretation	$\mathbf{H}/\Gamma(7)$	$\mathbf{H}/H^5(4-\sqrt{5})$
Cusps	24 cusps of $\Gamma(7)$	60 cusps of $\mathbf{H}/H^5(4-\sqrt{5})$

table 1.1

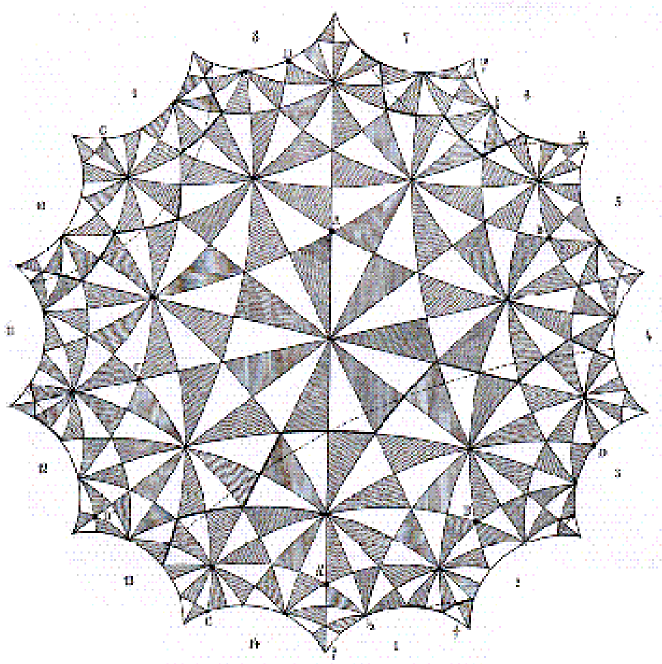
Here, $\mathbf{H}$ is the upper half-plane. The Hecke group H^5 is the Fuchsian group generated by $z \mapsto -1/z$ and $z \mapsto z + \lambda$, where λ is the golden ratio $\frac{1+\sqrt{5}}{2}$. By $(4-\sqrt{5})$ we mean the principal ideal of $\mathbf{Z}[\lambda]$ generated by $4-\sqrt{5}$.

The buckyball is just a truncated icosahedron. It is named after the architect Buckminster-Fuller, after some geodesic domes that he constructed. It has achieved fame because it is the shape of the carbon molecule C_{60}, which has been called the world's most beautiful molecule. [AW] It is also the most common shape of a football! A picture of the buckyball is given in Fig. 3 in section 8.

Part 1. $p = 7$.

2 The Klein quartic

The Klein quartic is the Riemann surface of the projective curve with equation $x^3y+y^3z+z^3x = 0$. This surface has genus 3. It has a special importance in Mathematics. In particular, it is the least genus surface S_3 for which the Hurwitz $84(g-1)$ bound is attained for the size of its automorphism group. [L1] In this case, $\mathrm{Aut}S_3 \cong$ PSL(2,7).



Identification
Table:
1–6
3–8
5–10
7–12
9–14
11–2
13–4

Figure 1: The Klein quartic

3 Finite planes and biplanes

The material in this section is taken from [P]. A *finite (projective) plane* is a finite collection of points and lines that satisfy the following conditions:

- Two distinct points lie on a unique line,
- There exist four points of which no three are incident with the same line.

It is well-known that that there is an integer $n > 1$ such that every line contains n points and every point is contained in n lines. Such a projective plane is said to have order n. A projective plane of order n has n^2+n+1 points and n^2+n+1 lines. The smallest example is the well-known *Fano plane* of order 2. This has seven points and seven lines and is just the projective plane over the field $\mathbf{F}_2$ with two elements. If the seven points are $1, 2, \ldots, 7$ the lines are given by the rows of the following table

1	2	4
2	3	5
3	4	6
4	5	7
5	6	1
6	7	2
7	1	3

table 3.1

Thus the first row are the quadratic residues mod 7, while the others are obtained by adding $1, 2, \cdots, 6$

The collineation group of the Fano plane is PGL(3,2)≅ PSL(2,7).

Biplanes A *biplane* is a system of points and lines obeying the axioms:

- B1 Two distinct points are contained in two distinct lines,
- B2 Two distinct lines intersect in two distinct points.

If a biplane has k points on each line then we say that its *order* is $k - 2$.

The biplane of order 2 is the *complement* of the Fano plane. This is the biplane whose points are the points of the Fano plane and whose lines are the complement of the lines of the Fano plane. For example a line in this biplane is 3, 5, 6, 7, which is complementary to the line 1, 2, 4 of the Fano plane. We will return to this idea later when we discuss the imbedding of the Fano plane in Klein's surface. The biplane of order 3 occurs in the discussion of the Buckyball, as noticed by Kostant. [K]. It is constructed in an analogous way as we constructed the Fano plane. Our points are $1, 2, \ldots, 11$ and our first line contains the quadratic residues mod 11, namely $1, 3, 4, 5, 9$. The other lines are found by adding $1, 2, \ldots, 11$. We thus get the lines

1	3	4	5	9
2	4	5	6	10
3	5	6	7	11
4	6	7	8	1
5	7	8	9	2
6	8	9	10	3
7	9	10	11	4
8	10	11	1	5
9	11	1	2	6
10	1	2	3	7
11	2	3	4	8

table 3.2

4 Imbedding graphs and hypergraphs into Riemann surfaces,

We now discuss the imbedding of the Fano plane into Klein's surface. This will be as a *hypermap*. We now briefly discuss the theory of maps and hypermaps.

There is a well-established theory of graph and hypergraph imbeddings into Riemann surfaces to form maps and hypermaps. A map is an imbedding of a graph $\mathcal{G}$ into a surface X such that the components of $X \setminus \mathcal{G}$ are polygonal 2-cells. It was observed in [JS1] that this makes X into a Riemann surface. In this way a map defines a complex algebraic curve and Grothendieck [G] noticed that Belyi's Theorem implies that this curve is defined over the field of algebraic numbers. If m is the lcm of the vertex valencies and n is the lcm of the face valencies then the map is said to be of *type* (m, n). The way of describing maps in [JS1] is to consider the universal tessellation $\hat{\mathcal{M}}(m, n)$ of type (m, n), that is the tessellation of a simply connected Riemann surface $\mathcal{U}$ (usually the hyperbolic plane) by triangles with angles $(\pi/2, \pi/m, \pi/n)$. We consider the group Γ^* generated by reflections in the sides of a triangle and let Γ denote the subgroup of index 2 consisting of sense preserving transformations. Now let M be a subgroup of finite index in Γ. Then M is a Fuchsian group that maps $\hat{\mathcal{M}}(m, n)$ to itself and thus $\hat{\mathcal{M}}(m, n)/M$ is a map on the Riemann surface $\mathcal{U}/M$. All maps of type $(m.n)$ can be constructed in this way.

Basically, a hypergraph is rather like a graph, except that an edge may contain any number of vertices. A precise definition is as follows: A hypergraph $\mathbf{H}$ is a pair consisting of a vertex set $V(\mathbf{H})$ together with a subset of edges $E(\mathbf{H})$ whose elements are subsets of $V(\mathbf{H})$. Thus we can speak of a vertex lying on an edge. The *valency* of an edge e is the number of vertices lying on e and the valency of a vertex v is the number of edges in which v lies. If every vertex has valency 1 or 2 then the hypergraph is a *graph*. A hypermap is an imbedding of a hypergraph $\mathcal{H}$ in a surface such that the components of $X \setminus \mathcal{H}$ are polygonal 2-cells. A hypermap has type (l, m, n) if l is the lcm of the edge valencies, m is the lcm of the vertex valencies and n is the lcm of the face valencies, The theory of hypermaps follows almost word by word the theory of maps, except that we now use triangles with angles $(\pi/l, \pi/m, \pi/n)$. [C], [CS], [JS2]. A hypermap is basically what Grothendieck [G], calls a dessin d'enfant (or just dessin) and then maps are the so called clean dessins.

There are many ways of drawing pictures of hypermaps, each with its own advantages. For our main application, we shall use the Cori representation.

The Cori representation. [C], [CS] Let $\mathcal{X}$ be a compact orientable surface. A hypermap $\mathcal{H}$ on $\mathcal{X}$ is a triple $(\mathcal{X}, S, A)$, where S,A are closed subsets of $\mathcal{X}$ such that:

1. $B = S \cap A$ is a non-empty finite set;
2. $S \cup A$ is connected;
3. Each component of S and each component of A is homeomorphic to a closed disc;
4. Each component of $\mathcal{X} \setminus (S \cup A)$ is homeomorphic to an open disc.

The elements of S are called *hypervertices* the elements of A are called *hyperedges* and the elements of B are called *hypervertices.*

In this paper a crucial example of a hypermap is the embedding of the Fano plane in Klein's surface. The Fano plane is the projective plane defined over a field of order 2. Thus each point has homogeneous coordinates of the form (a, b, c) with $a, b, c \in \mathrm{GF}(2)$ except that we do not allow $(0, 0, 0)$. Thus the Fano plane has 7 points and 7 lines, and its collineation group is PGL(3,2)$\cong$ PSL$(2, 7)$, the simple group of order 168.

The Cori representation of the Fano plane imbedded in Klein's surface was found in [S2] in 1985. and is pictured here as Fig.2 in section 8.

Another way of embedding the Fano plane is to consider the order two-biplane as the complement of the Fano plane as described in section 3, as we now show.

5 Imbedding the biplane of order 2 in Klein's surface.

We first find an imbedding of S_4 in PSL(2,7). A presentation of S_4 is

$$\langle x, y | x^2 = y^3 = (xy)^4 = 1 \rangle$$

We use the well-known facts that elements of order 2 have trace 0, elements of order 3 have trace ± 1, and elements of order 4 have trace whose square is equal to 2. One solution is

$$x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad y = \begin{pmatrix} 5 & 0 \\ 3 & 3 \end{pmatrix}$$

and then

$$xy = \begin{pmatrix} 3 & 3 \\ 2 & 0 \end{pmatrix}$$

which has order 4 as required. We now list the conjugacy classes of cyclic groups of order 3 in S_4 . It is easily seen that these are given by the cyclic groups generated by y, xyx, $xyxy^{-1}$ and $y^{-1}xyx$. Each of these act on the projective line $\mathbf{GF}_7 \cup \{\infty\}$. Each of these cyclic groups is the stabilizer of two points on when acting on $\mathbf{GF}_7 \cup \{\infty\}$. These are $\{0, 3\}$, $\{2, \infty\}$, $\{1, 6\}$, $\{4, 5\}$. We can also see that two fixed points determine the element of order 3 in S_4 and so the four pairs of elements of $\mathbf{GF}_7 \cup \{\infty\}$determine an embedding of S_4 in PSL(2, 7). Thus we can determine all the embeddings of S_4 in PSL(2, 7) by listing the fixed points of the 4 elements of order 3. There are two conjugacy classes of subgroups isomorphic to S_4 in PSL(2, 7). We call these two classes, P and L.

SUBGROUPS OF $PSL(2, 7)$ OF ORDER 24
IN TWO CONJUGACY CLASSES

Class P	Class L
$\Big\{\{4,5\}\{6,1\}\{0,3\}\{2,\infty\}\Big\}$	$\Big\{\{4,5\}\{6,2\}\{0,\infty\}\{1,3\}\Big\}$
$\Big\{\{5,6\}\{0,2\}\{1,4\}\{3,\infty\}\Big\}$	$\Big\{\{5,6\}\{0,3\}\{1,\infty\}\{2,4\}\Big\}$
$\Big\{\{6,0\}\{1,3\}\{2,5\}\{4,\infty\}\Big\}$	$\Big\{\{6,0\}\{1,4\}\{2,\infty\}\{3,5\}\Big\}$
$\Big\{\{0,1\}\{2,4\}\{3,6\}\{5,\infty\}\Big\}$	$\Big\{\{0,1\}\{2,5\}\{3,\infty\}\{4,6\}\Big\}$
$\Big\{\{1,2\}\{3,5\}\{4,0\}\{6,\infty\}\Big\}$	$\Big\{\{1,2\}\{3,6\}\{4,\infty\}\{5,0\}\Big\}$
$\Big\{\{2,3\}\{4,6\}\{5,1\}\{0,\infty\}\Big\}$	$\Big\{\{2,3\}\{4,0\}\{5,\infty\}\{6,1\}\Big\}$
$\Big\{\{3,4\}\{5,0\}\{6,2\}\{1,\infty\}\Big\}$	$\Big\{\{3,4\}\{5,1\}\{6,\infty\}\{0,2\}\Big\}$

Here the class P consists of one conjugacy class of subgroups of PSL(2, 7) isomorphic to S_4 and and the class L consists of the other conjugacy class, We denote the elements of P as $P_1, P_2 \ldots, P_7$ and the elements of L as as $L_1, L_2 \ldots, L_7$.

We shall now show how the biplane of order 2 can be constructed from the two conjugacy classes of subgroups isomorphic to S_4 in PSL(2,7), given in 4.1 above. Regard the first set of conjugacy classes as the points and the second set as the lines. giving 7 points and 7 lines. If a pair of fixed points of P_i coincides with a pair of fixed points of L_j then we write $P_i \cap L_j \neq \emptyset$ and we regard this as defining an incidence and so we can form a 7×7 incidence table.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

We see that any two lines intersect in two points and as there are 4 1's in each column, and row, we have a biplane of order 2. Note also that the "complementary" geometry [P] obtained by considering the 3 zeros in each row and column gives us a Fano plane structure. (For the lines are 4,5,7; 1,5,6; 2,6,7; 1,3,7, 1,2,4; 2,3,5; 3,4,6, that is the lines of the Fano plane.) This is a different way of embedding the Fano plane in Klein's quartic to that described in [S2].

6 The imbeddings of the truncated cubes in Klein's surface

A subgroup of PSL(2, 7) isomorphic to S_4 acts regularly (freely and transitively) on the 24 points stabilized by a C_7 on $\mathcal{K}$, Klein's surface. These 24 points are well-known to be the Weierstrass points of $\mathcal{K}$. As PSL(2, 7) contains 28 cyclic subgroups of order 3 and there are 168/3=56 points of the Klein map of valency 3, each cyclic group of order 3 fixes 2 points of $\mathcal{K}$. As each S_4 contains 4 C_3's this gives 8 fixed points. An example, the points $A, A^{'}, B, B^{'}, C, C^{'}, D, D^{'}$ in Klein's main figure in [L1], p320 are 8 such points, where $A, A^{'}$ etc. are two of the fixed points of one of the C_3s in S_4. These 8 points can be thought of topologically as the vertices of an inscribed cube, (or more precisely as Klein puts it the surface $\mathcal{K}$ can be stretched symmetrically onto a sphere so that the 8 points $A, A^{'}$, etc coincide with the vertices of an inscribed cube,)

Now from Klein's main figure in [L1], each point fixed by an element of order 3 is the centre of a hyperbolic equilateral whose vertices have valency 7. This gives us our $3 \times 8 = 24$ vertices of valency 7 which form the vertices of the truncation of the cube $A, A^{'}, B, B^{'}, C, C^{'}, D, D^{'}$. (Labelling in Klein's main figure.)

Finally, we want to represent the biplane geometrically on $\mathcal{K}$. First notice that each truncated cube corresponds to the S_4 in PSL(2, 7) which stablilizes it. Now there are 14 S_4's in PSL(2,

7) in two conjugacy classes of 7. Thus there are 2 classes of truncated cubes each containing 7 elements, Let us call the 7 truncated cubes in one class, points and the 7 truncated cubes of the other class lines. Each cube is determined by the 8 fixed points of the $4C_3$s which form the 8 vertices of the cube. The incidence pattern is then determined by the incidence matrix in the last section. Thus we get the biplane structure back again.

We summarise in

Theorem 1

(a) We consider the two conjugacy classes of 7 subgroups represented by points $P_1, P_2 \ldots, P_7$ and lines $L_1, L_2 \ldots L_7$ as described, With incidence as above we get a 2-biplane or Fano plane structure.

(b) (Geometric interpretation.) The Klein surface contains two classes of embedded truncated cubes. Their incidence pattern gives a 2-biplane or Fano plane structure on the Klein surface.

Klein's surface and triangle groups, Consider the (2,3,7) triangle group with presentation

$$< A, B, C | A^2 = B^3 = C^7 = ABC = 1 > .$$

There is a epimorphism $\Phi : (2,3,7) \to$PSL(2,7) defined by

$$\Phi(A) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Phi(B) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Phi(C) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

and by the Riemann-Hurwitz formula, KerΦ is a surface group of genus 3 which uniformizes the Klein surface $\mathcal{K}$. There is also an obvious homomorphism $\Psi : \Gamma \to$ PSL(2,7) defined by reduction mod 7, where Γ =PSL(2,Z) is the modular group and a homomorphism $\sigma : \Gamma \to (2,3,7)$ such that $\Phi \circ \sigma = \Psi$.

$$\begin{array}{ccccc}
\Gamma & \xrightarrow{\quad\sigma\quad} & [2,3,7] & \xrightarrow{\quad\Phi\quad} & PSL(2,7) \\
\uparrow 8 & & \uparrow 8 & & \uparrow 8 \\
\Gamma_0(7) & \longrightarrow & (0;3,3,7) & \longrightarrow & \text{Aff}(1,7) \\
\uparrow 3 & & \uparrow 3 & & \uparrow 3 \\
\Gamma_1(7) & \longrightarrow & (0;7,7,7) & \longrightarrow & C_7^\infty \\
\uparrow 7 & & \uparrow 7 & & \uparrow 7 \\
\Gamma(7) & \longrightarrow & K & \longrightarrow & \{1\}
\end{array}$$

Where $\frac{\mathcal{U}}{\Gamma(7)}$ is a Riemann surface of genus $g = 3$ with 24 punctures. This is because an element of order 7 in PSL(2, 7) has 168/7=24 fixed points.

In the above diagram the groups $\Gamma_0(7)$ and $\Gamma_1(7)$ are the usual congruence subgroups of Γ, i.e. $\Gamma_0(7) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \in \Gamma, c \equiv 0\text{mod}7\}$, $\Gamma_1(7) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, a \equiv d \equiv 1\text{mod}7, \text{c} \equiv 0\text{mod}7\}$.

7 The cusps of $\Gamma(7)$

The cusps of Γ, i.e.the parabolic fixed ponts of Γ is the set $\mathbf{Q} \cup \{\infty\}$. It is convenient to write $\frac{a}{c}$ as (a, c) so that $\infty = (1, 0)$.

The cusps are of the form $\{(a, c)|a, c \in \mathbf{Z}_n \text{ such that } (a, c, n) = 1\}/\sim$, where $(a, c) \sim (n-a, n-c)$. We can then show that the number of cusps of $\Gamma(n)$ is

$$\frac{n^2}{2}\prod_{p|n}(1-\frac{1}{p^2})$$

where the product is taken over all positive divisors p of n In particular, $\Gamma(7)$ has 24 cusps. These correspond to the 24 points of valency 7 on the Klein surface.

By choosing representatives of the cusps, we can show that the cusps of $\Gamma(7)$ are $(0, 1)$, $(1, 1) \ldots, (6, 1)$ $(1, 3), (4, 3) \ldots, (19, 3)$, $(1, 2), (3, 2), \ldots .(13, 2)$. The other 3 cusps are ∞, $(2, 7)$ and $(3, 7)$ As $(k, 7)$ is equivalent to $(k, 0)$ for $k = 2, 3$) we can regard these last 3 cusps as the infinite cusps, while the other 21 cusps are the finite cusps, As $z \mapsto z + 1$ fixes the 3 infinite cusps, we can regard them as the hypervertex centre, the hyperedge centre and the hyperface centre.

We now want to give a model of the Fano plane using the 21 finite cusps above. If we refer to the drawing of the Fano plane (Fig. 2, in section 8), we note the cusps are at the intersection points (which are hypervertices) of hyperfaces (lighly coloured triangles) and hypereges (dark triangles). These points are called *brins* by Cori, [C]. We define two cusps $\frac{a}{b}$ and $\frac{c}{d}$ to be *adjacent* if $ad - bc \equiv \pm 1 \bmod 7$. From the picture in section 4 we see that every cusp should be a 4-valent vertex. Let's see what is joined to 0=$\frac{0}{1}$. The hyperfaces and hyperedges are triangles. We note that all triangles have the form (a, b) (c, d), $(a + c, b + d)$ or we can replace the last vertex by $(a - c, b - d)$. We now note that we cannot have $(0, 1)$ adjacent to $(1, 1)$ for then one of the triangles would have ∞ as a cusp which is impossible as ∞ is not one of the 21 cusps of the Fano hypermap. As every flag is a 4-valent vertex the only flags adjacent to 0=(0,1) are (1,2),(1,3) (-1,2) and (-1,3) Note that (-1,2)=(13,2) and (-1,3)=(13,3). By continuing in this way we can label all the flags by rationals as illustrated in Fig. 2) We summarise in

Theorem 2. We can build a model of the Fano plane with the 21 finite cusps represented by brins. Two flags $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent if and only if $ad - bc \equiv \pm 1 \bmod 7$.

A picture of the Fano plane together with the cusps is given in Fig. 2.

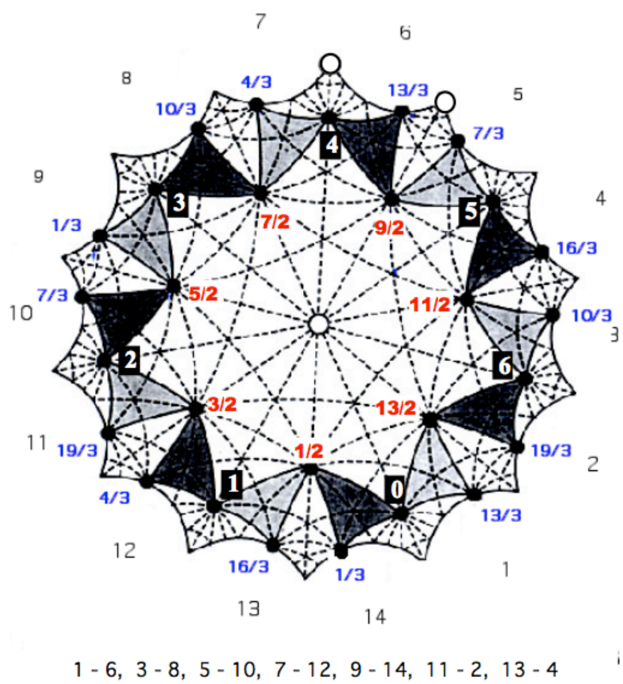


Figure 2: The fano plane.

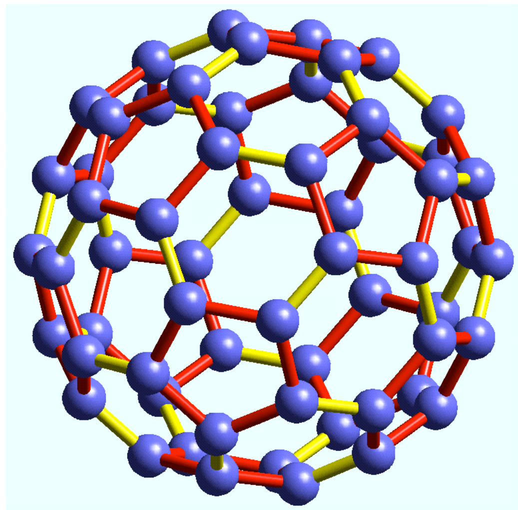


Figure 3: The buckyball.

Part 2, $p = 11$

8 H^5 and PSL(2,11)

In part 1 we constructed Klein's quartic by considering a homomorphism $\alpha : (2,3,7) \to$ PSL(2,7) whose kernel is K where $\mathbf{H}/K$ is the Riemann surface of Klein's quartic. We now consider a homomorphism $\phi : (2,5,11) \to$ PSL(2,11). If (2,5,11) $=< A, B, C | A^2 = B^5 = C^{11} = I >$, then ϕ is defined by

$$\phi(A) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \phi(B) = \begin{pmatrix} 0 & 1 \\ -1 & 8 \end{pmatrix}, \phi(C) = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$$

where the matrices represent the corresponding Möbius transformations and we reduce the elements of the matrices mod 11. This homomorphism is smooth (it preserves the orders of elements of finite order) and so the kernel is a surface group. As |PSL(2,11)| = 660, we can use the Riemann-Hurwitz formula to compute the genus of the kernel.

$$2g-2=660(1-\frac{1}{2}-\frac{1}{5}-\frac{1}{11})$$

giving $g=70$

Let H^5 denote the Hecke group with generators

$$\hat{A}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \hat{B}=\begin{pmatrix}0 & -1\\ 1 & \lambda\end{pmatrix}$$

where λ is the golden ratio as defined in the introduction. Group theoretically, we have

$$H^5 \equiv C_2 * C_5.$$

(Note that $\hat{B}$ has order 5.)

Now for an ideal $I \subseteq \mathbf{Z}[\lambda]$ we let $H_0^5(I) = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in H^5$ such that $c \equiv 0 \bmod I\}$ and $H_1^5(I) = \{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in H^5$ such that $a \equiv d \equiv 0 \bmod I$, $c \equiv 0 \bmod I\}$.

We clearly have a homomorphism $\psi : H^5 \to (2,5,11)$. Let Λ denote the kernel of ψ. Now by [LLT] $\frac{H^5}{H^5(4-\sqrt{5})}$ is isomorphic to PSL(2, 11). Now from the character table for PSL(2, 11) given in the 'atlas' on page 7 and from the formula given in [S1] we find that there are precisely 2 epimorphisms from H^5 to PSL(2, 11). (I would like to thank Gareth Jones for showing me this argument,) As $\frac{H^5}{H^5(4-\sqrt{5})}$ and $\frac{H^5}{H^5(4+\sqrt{5})}$ are images of H^5 so we have constructed these homomorphisms. We might as well use the epimorphism above induced from the epimorphism $H^5 \to \frac{H^5}{H^5(4-\sqrt{5})}$. Note that both the ideals $(4 \pm \sqrt{5})$ have norm 11. We can express everything above using the diagram

$$\begin{array}{ccccc}
H^5 & \xrightarrow{\psi} & [2,5,11] & \xrightarrow{\phi} & PSL(2,11)\\
\uparrow 12 & & \uparrow 12 & & \uparrow 12\\
H_0^5(4-\sqrt{5}) & \longrightarrow & (1;5,5,11) & \longrightarrow & \mathrm{Aff}(1,11)\\
\uparrow 5 & & \uparrow 5 & & \uparrow 5\\
H_1^5(4-\sqrt{5}) & \longrightarrow & (5;11^5) & \longrightarrow & C_{11}^{\infty}\\
\uparrow 11 & & \uparrow 11 & & \uparrow 11\\
H^5(4-\sqrt{5}) & \longrightarrow & \Lambda & \longrightarrow & \{1\}
\end{array}$$

Definition: We shall sometimes call $\mathcal{U}/\Lambda$ the *Buckyball* surface.

The Buckyball surface is also the compactification of $\mathcal{U}^*/H^5(4-\sqrt{5})$ where $\mathcal{U}^*$ is the union of $\mathcal{U}$ with the cusps of $\mathbf{Z}(\lambda)$.

Lemma 1 *There are 60 points on $\mathcal{U}/\Lambda$ whose stabilizer is isomorphic to C_{11}. Each element of order 11 fixes 5 points of $\mathcal{U}/\Lambda$.*

Proof. We can decompose $\mathcal{U}/\Lambda$ into triangles with angles $\frac{\pi}{2}.\frac{\pi}{5}, \frac{\pi}{11}$. By the Gauss-Bonnet Theorem the hyperbolic area of such a triangle is $\frac{23\pi}{110}$. The hyperbolic area of $\mathcal{U}/\Lambda$ is $2\pi(2\times 70-2) = 276\pi$. Thus the number of such triangles is $276\pi/(\frac{23\pi}{110}) = 1320$. Now draw another triangle obtained by reflecting one of our triangles through the line from the $\frac{\pi}{2}$ angle to the $\frac{\pi}{11}$ angle to get 660 triangles with angles $\frac{\pi}{5}, \frac{\pi}{5}$, and $\frac{2\pi}{11}$. The points of $\mathcal{U}/\Lambda$ fixed by a cyclic group of order 11 are at the vertices of these triangles with an angle $\frac{2\pi}{11}$, and so there number is 60. By the Sylow Theorems the number of C_{11}'s in PSL(2,11) is 12 so that each C_{11} fixes 5 points.

9 A^5 and PSL(2,11)

The group PSL(2, 11) contains as its largest subgroups two conjugacy classes of subgroups isomorphic to A_5. Each class contains 11 subgroups. The other large subgroup is the stabilizer of a point of $\mathbf{GF}_{11} \cup \{\infty\}$. This has order 55, and there is just one conjugacy class containing 12 subgroups. Now each subgroup isomorphic to A_5 contains, by the Sylow Theorems 6 subgroups of order 5. Now each C_5 <PSL(2,11) can be labelled with its pair of fixed points in $\mathbf{GF}_{11}$. and so (following the same idea as for PSL(2, 7)) we can characterize every $A_5 <$ PSL(2, 11) by a sextuplet of pairs of points.

Example. Consider the subgroup of PSL(2, 11) generated by the matrices

$$a = \begin{pmatrix} 9 & 0 \\ 6 & 5 \end{pmatrix} \text{ and } b = \begin{pmatrix} 9 & 10 \\ 0 & 5 \end{pmatrix}$$

These matrices, having the same eigenvalues can be shown to be conjugate in PSL(2,11), (or perform a direct calculation.)

The fixed points of a when acting on $\mathbf{GF}_{11}$ are 0 and 8 while the fixed points of b are 3 and ∞ and so we can label $< a >= C_5^{0,8} = \{0, 8\}$ and $b = C_5^{3,\infty} = \{3, \infty\}$.

Now by direct calculation ab^2 has order 2 and ab has order 3. Also $(ab)^{-1}ab^2$ has order 5. Thus the group generated by a and b is the (2,3,5) triangle group which is isomorphic to A_5.

By the Sylow Theorems there is one conjugacy class of C_5's in PSL(2,11) containing 6 subgroups. Generators for these subgroups are b, a, bab^{-1}, b^2ab^{-2}, b^3ab^{-3}, b^4ab^{-4}. As gbg^{-1} fixes $g(3)$ and $g(\infty)$, we find that the pairs of fixed points of the conjugates of b are $\{3, \infty\}$, $\{8, 0\}$, $\{1, 2\}$, $\{6, 10\}$, $\{4, 9\}$, $\{5, 7\}$.

As in section 5, for the seven imbeddings of S_4 in PSL(2, 7) we can use all the pairs of fixed points to determine the eleven imbeddings of A_5 in PSL(2, 11). The pairs of points in the eleven conjugacy classes are given in the following table.

SUBGROUPS OF $PSL(2,11)$ OF ORDER 60 IN TWO CONJUGACY CLASSES	
Class P	Class L
$\Big\{\{0,1\}\{4,6\}\{7,10\}\{5,9\}\{3,8\}\{2,\infty\}\Big\}$	$\Big\{\{0,1\}\{6,8\}\{2,5\}\{3,7\}\{4,9\}\{10,\infty\}\Big\}$
$\Big\{\{1,2\}\{5,7\}\{8,0\}\{6,10\}\{4,9\}\{3,\infty\}\Big\}$	$\Big\{\{1,2\}\{7,9\}\{3,6\}\{4,8\}\{5,10\}\{0,\infty\}\Big\}$
$\Big\{\{2,3\}\{6,8\}\{9,1\}\{7,0\}\{5,10\}\{4,\infty\}\Big\}$	$\Big\{\{2,3\}\{8,10\}\{4,7\}\{5,9\}\{6,0\}\{1,\infty\}\Big\}$
$\Big\{\{3,4\}\{7,9\}\{10,2\}\{8,1\}\{6,0\}\{5,\infty\}\Big\}$	$\Big\{\{3,4\}\{9,0\}\{5,8\}\{6,10\}\{7,1\}\{2,\infty\}\Big\}$
$\Big\{\{4,5\}\{8,10\}\{0,3\}\{9,2\}\{7,1\}\{6,\infty\}\Big\}$	$\Big\{\{4,5\}\{10,1\}\{6,9\}\{7,0\}\{8,2\}\{3,\infty\}\Big\}$
$\Big\{\{5,6\}\{9,0\}\{1,4\}\{10,3\}\{8,2\}\{7,\infty\}\Big\}$	$\Big\{\{5,6\}\{0,2\}\{7,10\}\{8,1\}\{9,3\}\{4,\infty\}\Big\}$
$\Big\{\{6,7\}\{10,1\}\{2,5\}\{0,4\}\{9,3\}\{8,\infty\}\Big\}$	$\Big\{\{6,7\}\{1,3\}\{8,0\}\{9,2\}\{10,4\}\{5,\infty\}\Big\}$
$\Big\{\{7,8\}\{0,2\}\{3,6\}\{1,5\}\{10,4\}\{9,\infty\}\Big\}$	$\Big\{\{7,8\}\{2,4\}\{9,1\}\{10,3\}\{0,5\}\{6,\infty\}\Big\}$
$\Big\{\{8,9\}\{1,3\}\{4,7\}\{2,6\}\{0,5\}\{10,\infty\}\Big\}$	$\Big\{\{8,9\}\{3,5\}\{10,2\}\{0,4\}\{1,6\}\{7,\infty\}\Big\}$
$\Big\{\{9,10\}\{2,4\}\{5,8\}\{3,7\}\{1,6\}\{0,\infty\}\Big\}$	$\Big\{\{9,10\}\{4,6\}\{0,3\}\{1,5\}\{2,7\}\{8,\infty\}\Big\}$
$\Big\{\{10,0\}\{3,5\}\{6,9\}\{4,8\}\{2,7\}\{1,\infty\}\Big\}$	$\Big\{\{10,0\}\{5,7\}\{1,4\}\{2,6\}\{3,8\}\{9,\infty\}\Big\}$

Again, we regard the set P to be the set of points and the set L to be the set of lines. To understand the geometry that we obtain we set up an incidence table as in section 5.

$$\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}$$

This matrix is interesting in that we can find a model of the order 3 biplane by examining the positions of the zeros. These are in positions 2,5,7,8,9 in the first row. If we regard this as a line then all lines are

$$\begin{pmatrix} 2 & 5 & 7 & 8 & 9 \\ 3 & 6 & 8 & 9 & 10 \\ 4 & 7 & 9 & 10 & 11 \\ 1 & 5 & 8 & 10 & 11 \\ 1 & 2 & 6 & 9 & 11 \\ 1 & 2 & 3 & 7 & 10 \\ 2 & 3 & 4 & 8 & 11 \\ 1 & 3 & 4 & 5 & 9 \\ 2 & 4 & 5 & 6 & 10 \\ 3 & 5 & 6 & 7 & 11 \\ 1 & 4 & 6 & 7 & 8 \end{pmatrix}$$

Notice that the 8th row is 1,3,4,5,9. These are just the quadratic residues mod 11. Regarding the points as elements of the finite field on 11 points, (with 11=0) we see that the other lines are just translations of the 8th row by $1, 2, \ldots, 10$.

10 The embedding of the Buckyballs in $\mathcal{U}/$Ł.

By Lemma 1, there are 60 points on the surface $\mathcal{U}/\Lambda$.whose stabilizer is isomorphic to C_{11} and each such element fixes 5 points of the surface, (By a result of Lewittes[L2], all these points are Weierstrass points but there must be other Weierstrass points as a Riemann surface of genus g contains at least $2g+2$ Weierstrass points.) Now by [D], PSL(2, 11) contains 66 subgroups of order 5, and as there are 660/5 points of valency 5 on the corresponding map each of the subgroup of order 5 fixes 2 points on the surface $\mathcal{U}/\Lambda$. As each $A_5 < PSL(2,11)$ contains 6 subgroups of order 5, we have 12 fixed points of the of $\mathcal{U}/\Lambda$ which are fixed by the A_5. As each C_5 has two fixed points, these fixed points are naturally paired. We can think of these points as forming a diagonal of an embedded icosahedron. By considering the tessellation formed by $\frac{\pi}{2}.\frac{\pi}{5}, \frac{\pi}{11}$ triangles on these triangles each point fixed by a C_5 lies at the centre of a regular pentagon whose vertices lie in the set of 60 points stabilized by a C_5. These 60 points then form the truncation of the icosahedron and so they form a buckyball. .

As there are two conjugacy classes of A_5s in PSL(2, 11) we can build a geometry, where one class comes from the fixed points of the conjugates of elements of one conjugacy class and the other from the other conjugacy class as we did in section 5. We get the same structure as in section 9, so that the embeddings of the Buckyballs in $\mathcal{U}/\Lambda$ give rise to a 3- biplane structure. .

We can now generalize Theorems 1 and 2 to the case $p = 11$.

Theorem 3(a). *We consider the two conjugacy classes of 11 subgroups represented by the classes P and L above. With incidence as defined we get a 3-biplane structure.*

3(b) The Buckyball surface contains two classes of embedded buckyballs. Their incidence pattern gives a 3-biplane structure.

We note that unlike the Fano plane (or 2-biplane) structure on Kleins surface, the 3-biplane structure does not give a hypermap, What we have is a conformal embedding in that a vertex of valency v gives v rays intersecting at angle $2\pi/v$.

11 The 3-biplane structure on the cusps of H^5.

Note that the cusps of H^5 are the same as the cusps of $H^5(4-\sqrt{5})$, as the latter has finite index in H^5. However, the number of equivalence classes of cusps is different. For H^5 all cusps are real numbers of the form $a + b\sqrt{5}$, (a,b rational), and there is just one class of cusps. However we have seen that for $H^5(4 - \sqrt{5})$ there are 60 classes of cusps. In section 7, we showed that the 24 cusps of $\Gamma(7)$ could be written as 21 cusps +3 cusps. (The 21 corresponding to the cusps of the Fano plane.) Here, we will write 60=55+5, the 55 corresponding to the points of the 3-biplane, and 5 corresponding to the "infinite cusps" $\infty, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}$. For $\Gamma(7)$ we had a triangular structure. For $H^5(4 - \sqrt{5})$, our guess is that we have a pentagonal structure. So we look for all solutions of $(a + b\sqrt{5})^5 \equiv 1 \bmod 11$, We find all solutions by MAPLE and they are listed as follows. The value of a is listed in the first column on the left and the 5 values of b are in columns 2 to 6. Thus, for example, if $a = 1$ then $b = 9, 0, 1, 2$ or 6.

0	1	3	4	5	9
7	2	4	5	6	10
3	3	5	6	7	0
10	4	6	7	8	1
6	5	7	8	9	2
2	6	8	9	10	3
9	7	9	10	0	4
5	8	10	0	1	5
1	9	0	1	2	6
8	10	1	2	3	7
4	0	2	3	4	8

We write the table in this way because it then becomes apparent that the solutions form the 3-biplane with 55 points. In this way we see that the cusps of $H^5(4 - \sqrt{5})$ can be given the structure of a 3-biplane. The next result which follows from the above calculation generalizes Theorem 2 and is stated here for convenience.

Theorem 4. *There is a 3-biplane structure on the cusps of* $H^5(4-\sqrt{5})$.

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